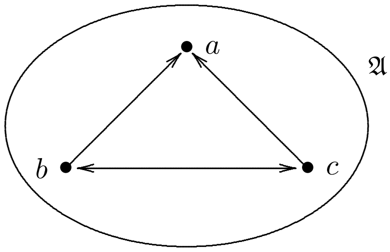
Set Theory – Spring 2003 – Homework 1

Name:
Due January 31

1. Which ZF axioms does the model $\mathfrak{A} = (A, E)$? satisfy? Explain four of them.



2. Let $\mathfrak{A} = \{X \subset \omega | X \text{ is finite of cofinite}\}$, and let $\preceq$ be the following binary relation on $\mathfrak{A}$:

$$X \preceq Y \Leftrightarrow \begin{cases} \text{either } X = Y \\ \text{or } \#(X) |_{\neq} \#(Y) \in \omega \\ \text{or } \#(\omega \setminus Y) |_{\neq} \#(\omega \setminus X) \in \omega \end{cases}$$

   (a) Prove that $\preceq$ orders $\mathfrak{A}$ in a non total way.

   (b) For $\mathfrak{B} = \{X \subset \omega | \#(X) \in \omega\}$ and $\mathfrak{C} = \{X \subset \omega | \#(\omega \setminus X) \text{ is a power of } 2\}$, compute $\max \mathfrak{B}$, $\min \mathfrak{C}$, Minim $\mathfrak{B}$, $\check{\mathfrak{C}}$, $\sup \mathfrak{B}$, where Minim $\mathfrak{B}$ denotes the set of minimal elements of $\mathfrak{B}$ and $\check{\mathfrak{C}}$ denotes the set of upper bounds of $\mathfrak{C}$.

3. In $L^1 = \{f : [0, 1] \to \mathbb{R} | \int_0^1 f dx \in \mathbb{R}\}$, consider the equivalence relation

$$fEg \Leftrightarrow \int_0^1 (f - g) dx = 0.$$

   (a) Find a **continuous** function in the equivalence class of $\chi_{\mathbb{Q}}$ (the characteristic function of $\mathbb{Q}$).

   (b) Is it true that if f is in the class of the constant function of value 1, then it must be positive?

   (c) If we change the last 0 in the definition of E to some $\epsilon > 0$, do we still have an equivalence relation?

4. What is wrong in the following proof by induction?
Theorem: all elements of any set are the same.
Proof: Induction on the size of the set. Clearly true for sets of size 1. Suppose the result holds for all sets of size n. Take a set a with $n+1$ elements, say $a = \{y_1, y_2, \ldots, y_{n-1}, y_n, y_{n+1}\}$. Then the sets $\{y_1, \ldots, y_{n-1}, y_n\}$ and $\{y_1, \ldots, y_{n-1}, y_{n+1}\}$ both have n elements, so (by induction hypothesis) all the y_i are equal to y_1.

5. Prove that if $|S| \geq 2$, then $|T| \leq |S^T|$.

6. Prove that $\mathfrak{A} = \langle \mathbb{R}, +, <, 0 \rangle$ is isomorphic to $\mathfrak{B} = \langle \mathbb{R}^+, \cdot, <, 1 \rangle$.

7. Prove that $]0, 4]$ is equipotent to $[1, 3]$.

8. Consider the plane $\mathbb{R}^2$, and the order relation

$$(x,y) \preceq (z,w) \Leftrightarrow \begin{cases} \text{either} & z = 0,\ w > 0,\ (x \neq 0 \text{ or } (x = 0 \text{ y } y < 0)) \\ \text{or} & x = 0,\ y < 0,\ (z \neq 0 \text{ or } (z = 0 \text{ y } w > 0)) \\ \text{or} & (x,y) = (z,w) \\ \text{or} & z \neq 0 \text{ and } x \neq 0 \text{ and } \frac{y}{x^2} < \frac{w}{z^2} \end{cases}$$

Find maximal elements, minimal elements, max, min, upper and lower bounds, sups and infs of S^1, $\mathbb{R} \times \{0\}$ and $\{(x,y) | 2x^2 + y^2 = 1\}$.