Set Theory – Spring 2003 – Homework 5

Name:

Due April 16

Consider the following operations ($\mathcal{F}$ is a class of sets of real numbers):

$$\sigma(\mathcal{F}) = \{\bigcup_{i<\omega} A_i | (A_i)_{i<\omega} \text{ is a sequence of elements of } \mathcal{F}\}$$

$$\delta(\mathcal{F}) = \{\bigcap_{i<\omega} A_i | (A_i)_{i<\omega} \text{ is a sequence of elements of } \mathcal{F}\}$$

(So, σ computes the family of all countable unions of members of $\mathcal{F}$ and δ computes the family of all countable intersections of members of $\mathcal{F}$.)

Let Σ_0 consist of all open sets of reals, and Π_0 of all closed sets of reals.

Consider the following transfinite inductive definition:

$\Sigma_{\alpha+1} := \sigma(\Pi_\alpha)$,

$\Delta_{\alpha+1} := \delta(\Sigma_\alpha)$,

$\Sigma_\mu := \bigcup_{i<\mu} \Sigma_i$ (μ limit ordinal),

$\Pi_\mu := \bigcup_{i<\mu} \Pi_i$ (μ limit ordinal).

(Sets belonging to any of the "levels" Π_α or Σ_α are called **Borel sets**. The whole structure is called the **Borel hierarchy** for reasons you will see in the questions.)

1. Find $A \in \Sigma_1 \setminus \Sigma_0$, $B \in \Pi_1 \setminus \Pi_0$. (Explain!)
2. Show that $\alpha < \beta$ implies $\Sigma_\alpha \subseteq \Sigma_\beta$. (The same holds for the Π hierarchy, but you do not have to write that proof!)

3. Prove that for all ordinal α, $\Pi_\alpha \subseteq \Sigma_{\alpha+1}$. (Again, the dual also holds: $\Sigma_\alpha \subseteq \Pi_{\alpha+1}$ for every α... do not write this proof).

So, by the previous, the hierarchy consists of 'interwoven' Π and Σ's.

$$\begin{array}{ccccccccccc}
\Sigma_0 & \rightarrow & \Sigma_1 & \rightarrow & \cdots & \rightarrow & \Sigma_\alpha & \rightarrow & \Sigma_{\alpha+1} & \rightarrow & \cdots \\
 & \searrow & & \searrow & \cdots & \searrow & & \searrow & & \searrow & \cdots \\
 & \nearrow & & \nearrow & \cdots & \nearrow & & \nearrow & & \nearrow & \cdots \\
\Pi_0 & \rightarrow & \Pi_1 & \rightarrow & \cdots & \rightarrow & \Pi_\alpha & \rightarrow & \Pi_{\alpha+1} & \rightarrow & \cdots
\end{array}$$

(The arrows mean $\subseteq$ above.) It is actually harder to prove that for $\alpha < \omega_1$ we have *strict* inclusions.

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4. **(This question is worth** 50% **of this homework!** Prove that the hierarchy 'stabilizes' at level ω_1:

$$\Sigma_{\omega_1+1} = \Sigma_{\omega_1} = \Pi_{\omega_1} = \Pi_{\omega_1+1}.$$

(Hint: use the fact that $\aleph_1$ is regular.)

Discussion 0.0.1 *1. It is harder to see that there are in $\mathbb{R}$ 'arbitrarily complex' Borel sets, below the level of complexity ω_1. For this we have to prove that the inclusions above are strict, all the way up to ω_1.*

2. The last question also implies that, although many sets in Analysis are Borel, there are 'very few' Borel sets, from a set-theoretical perspective: the first levels (open sets/closed sets) are of size continuum (by the density of the rationals), by induction on $\alpha < \omega_1$, each level is of size continuum, and there only ω_1 many levels.

3. Borel sets, although extremely important for analysis, are not the only ones studied there. An arbitrary continuous image of a Borel set may be outside of the Borel hierarchy...