Set Theory – Spring 2003 – Homework 4

Name:

Any five due March 31

1. Given any limit ordinal α, an α-sequence in a set X is a map from α into X (so ω-sequences are our old "sequences").
   (a) i. Find an **increasing** ω-sequence in $\mathbb{R}$.
       ii. Find an **increasing** $\omega + \omega$-sequence in $\mathbb{R}$.
       iii. Find an **increasing** ω^2-sequence in $\mathbb{R}$.
   (b) Find an **increasing** ε_0-sequence in $\mathbb{R}$.
   (c) Can you find an **increasing** ω_1-sequence in $\mathbb{R}$? (Write one such sequence if your answer is yes or explain **clearly** why not.)
   (d) Define $(X, \prec)$ such that
       i. $\prec$ is a complete dense linear ordering of X, and
       ii. There is an increasing ω_3-sequence in X.
2. Prove that if $X \subset \omega_\alpha$ is such that $|X| < \aleph_\alpha$, then $|\omega_\alpha \setminus X| = \aleph_\alpha$.
3. Prove that the Axiom of Choice implies the **Principle of Dependent Choices:** Any binary relation R on a nonempty M such that for each $x \in M$ there is $y \in M$ with xRy, then there is a sequence $\langle x_n | n < \omega\rangle$ such that $x_n R x_{n+1}$ for all $n < \omega$.
4. Show that there exist arbitrarily high ordinals α such that $\aleph_\alpha = \alpha$. Find the first such α.
5. Compute the cardinality of ω-dimensional Hilbert space
$$\mathcal{H} = \{f : \omega \to \mathbb{R} | \sum_{n<\omega} (f(n))^2 \text{ converges}\}$$
6. Given an ordinal α, let E_α be the set of all ω-sequences of ordinals $< \alpha$ that are eventually null, and let $\prec$ be the **inverse** lexicographical ordering on E_α. What ordinals can you embed in an increasing way into $(E_\omega, \prec)$? Into $(E_{\omega_1}, \prec)$?